Table 1 Deviation of internal predicted temperatures, $N = 2$

Time hr	(0.15, 0.3) $n = 5$	(0.15, 0.7) $n = 5$	(0.3, 0.15) $n = 5$	(0.7, 0.15) $n = 5$
0.10	2.12×10^{-2}	3.59×10^{-2}	8.45×10^{-2}	-4.40×10^{-2}
0.15	-0.12×10^{-2}	-0.29×10^{-2}	-1.03×10^{-2}	-0.43×10^{-2}
0.20	0.19×10^{-2}	0.22×10^{-2}	0.33×10^{-2}	-0.30×10^{-2}
0.25	0.09×10^{-2}	0.01×10^{-2}	-0.60×10^{-2}	0.16×10^{-2}
0.30	0.32×10^{-2}	0.43×10^{-2}	1.08×10^{-2}	-0.78×10^{-2}

at the corners of rectangular solid is an insufficient requirement, since additional information must be provided as to the behavior between the corners.

Should the corner solution prove to be unacceptable, then it is necessary to introduce additional perimeter thermocouple responses as a set of four, one for each face. As shown in Ref. 2, the original corner solution is amended to incorporate the new thermocouple inputs. The numerical results are then compared with the nonutilized perimeter positions, and the method can be repeated to embrace as many thermocouple inputs as desired. There is, however, a practical limitation. The related matrix equation for the enabling coefficients grows as $4(n)$ per set of thermocouple inputs. If N represents the number of matched points excluding the corners, the size of the involved matrix would be $4(n)(N)$. In other words, a two point match with a fifth degree approximation polynomial entails the solution of 40 equations with 40 unknowns. Furthermore, the sensitivity grows with the size of the matrix. A partial reduction of this effect may be achieved by re-arranging the coefficient matrix so that the diagonal elements contain the largest values. It is, however, recommended that the matrix size be kept as small as possible.

Tables 1–3 indicate the extrapolation results for a rectangular solid with sides (1 ft, π ft). Three of the rectangle's faces are maintained at zero temperature, and the remaining face, $x = 0$, varies linearly with time, i.e., $T(0, y, t) = t$. The thermocouple rectangle is a square of length 0.4 ft with the nearest corner located at (0.3 ft, 0.3 ft).

Table 1 represents a comparison of the predicted and sentinel traces at the four positions denoted by the braces. For a two point match coupled with a fifth degree approximation, the backward extrapolation is not successful for the earliest time increment, $t = 0.10$ hr. In the main, there is some increase in accuracy over the one point match shown in Ref. 2. On the basis of these results, backward prediction to the relevant faces should be satisfactory, except for, $t = 0.10$ hr. Table 2 is a compilation of the surface predicted values, and as anticipated good agreement occurs for all values except, $t = 0.10$ hr. A comparison of the deviations contained in Tables 1 and 2 reveals an unexpected result. Apparently, the magnitude of the deviation grows as the extrapolation distance increases. Table 3 presents the computational results for forward extrapolation. The preceding observations also apply to this situation. At the earliest time, forward extrapolation is unconvincing. The remaining tabulated results are quite good.

Table 2 Surface prediction temperatures, $N = 2$

Time hr	(0, 0.3) $n = 5$	(0, 0.7) $n = 5$	(0.3, 0) $n = 5$	(0.7, 0) $n = 5$
0.10	0.0646	0.0394	−0.1100	0.0584
0.15	0.1550	0.1539	0.0226	−0.0087
0.20	0.1988	0.1905	−0.0111	0.0098
0.25	0.2526	0.2459	0.0175	−0.0045
0.30	0.2948	0.2795	−0.0534	0.0350

Table 3 Deviation of prediction temperatures, $N = 2$

Time hr	(0.85, 0.3) $n = 5$	(0.85, 0.7) $n = 5$	(1.0, 0.3) $n = 5$	(1.0, 0.7) $n = 5$
0.10	-1.37×10^{-2}	-0.47×10^{-2}	-2.13×10^{-2}	-0.78×10^{-2}
0.15	0.20×10^{-2}	0.08×10^{-2}	0.54×10^{-2}	0.15×10^{-2}
0.20	0.00×10^{-2}	0.02×10^{-2}	0.06×10^{-2}	0.00×10^{-2}
0.25	0.12×10^{-2}	0.06×10^{-2}	-0.41×10^{-2}	0.11×10^{-2}
0.30	-0.05×10^{-2}	0.00×10^{-2}	0.16×10^{-2}	-0.09×10^{-2}

As previously mentioned, the coefficient matrix for the enabling function coefficients is large, 40 by 40, with elements that are very small. Accordingly, precise determination of the appropriate values for the first time interval is very difficult. Since the analytical solution should return the perimeter data inputs, the computations reveal this is generally so except at, $t = 0.10$ hr. For the initial time, the deviation may be as high as 200%; however the magnitude of the quantities involved are extremely small. It therefore follows that the prediction process will do well for longer times, since the error contribution from the inaccuracy associated with the initial time value is negligible.

In conclusion, the two-dimensional extrapolation method is considerably more sensitive than the one-dimensional case. It is advisable to position sentinel thermocouples whenever possible. By placing the thermocouple rectangle in close proximity to the region of extrapolation, error growth may be contained. The matrix size should be kept as small as possible. This can be achieved by using a smaller order approximation polynomial coupled with a minimum number of matched points. In situations where this is not possible, some sacrifice in accuracy must be accepted.

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Analytical Solution for Shock Wave Precursors

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I. Introduction

In the last few years the literature has contained several reports of experimental and theoretical investigations of the electron and excited state populations in shock wave precursors. Some of the theoretical investigations involve the complete non-equilibrium shock wave structure behind the shock[1–4] while others assume a blackbody emitter behind the shock.[5–7] Measurements of precursor populations have been reported by Weymann,[8] Holmes and Weymann,[9] Teshima et al.,[10] and Lederman and Wilson.[11] This Note presents a simple analytical solution for the precursor electron and excited state number density due to radiative processes and compares the results with

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numerical results including collisional deionization and de-excitation.

II. Mathematical Model

The precursor region in front of strong shock waves in atomic gases can be investigated by considering a model which consists of a three-level atom, a normal shock wave, and a one-dimensional geometry. Previous investigations of the precursor region have shown that the temperature and density are essentially constant.[1,2] Thus the precursor structure can be calculated by considering only the excitation and ionization rates. In the precursor the gas is cold so that the collisional excitation and ionization rates are negligible compared to their reverse rates. Thus, using the notation of Nelson,[1] the collisional rates in the precursor are written as follows:

a) electron-atom recombination to excited state

$$\bar{n}_{ea} = -l\bar{T}_{13}{}^{1.5} g_2(\bar{T}_{23}/2+1)\alpha^3/(l_{ea}{}^+ \bar{\rho}_{ion}) = -C_{ea}\alpha^3 \quad (1)$$

b) atom-atom recombination to excited state

$$\bar{n}_{aa} = -l\bar{T}_{13}{}^{1.5} g_2(\bar{T}_{23}/2+1)\alpha^2/(l_{aa}{}^+ \bar{\rho}_{ion}) = -C_{aa}\alpha^2 \quad (2)$$

c) electron-atom de-excitation to ground state

$$\bar{n}_{ea}{}^* = -lg_1(\bar{T}_{12}/2+1)\alpha\beta/(l_{ea}\, g_2) = -C_{ea}{}^*\alpha\beta \quad (3)$$

d) atom-atom de-excitation to ground state

$$\bar{n}_{aa}{}^* = -lg_1(\bar{T}_{12}/2+1)\beta/(l_{aa}\, g_2) = -C_{aa}{}^*\beta \quad (4)$$

The above equations are simplified from Eqs. (17–20) that appear in Ref. 1 because α and β are assumed to be much less than unity, and the temperature and density are constant and equal to unity. The radiative rates are reformulated from those that appear in Ref. 1 in order to take into account that fact that the line wing is very small near the excited state ionization edge. Thus, the excited state continuum and line wing radiation separate into two terms, one containing only the excited state continuum and the other containing only the line wing radiation. The ground state continuum and the line center radiation are exactly the same as formulated in Ref. 1. Using the same approximations as were used in the collisional rates, the radiative rates become

e) radiative production of electrons

$$\bar{n}_r = -\frac{\Gamma}{a}\frac{d}{d\eta}\{\bar{T}_{13}{}^2\bar{Q}_g(\tau_g)+\bar{T}_{23}{}^2\bar{Q}_s(\tau_s)\} \quad (5)$$

f) radiative production of excited states

$$\bar{n}_r{}^* = -\frac{\Gamma}{a}\frac{d}{d\eta}\{\bar{T}_{12}{}^2\bar{Q}_c(\tau_c)+\bar{T}_{12}{}^2\bar{Q}_w(\tau_w)+\bar{T}_{23}{}^2\bar{Q}_s(\tau_s)\} - l\beta/l_A \quad (6)$$

In Eqs. (1–6) Γ, a, $\bar{T}_{13}$, $\bar{T}_{12}$, $\bar{T}_{23}$, $\bar{Q}_s$, $\bar{Q}_c$, and $\bar{Q}_w$ are nondimensionalized by characteristic values in the hot gas behind the shock wave, while the remaining variables are nondimensionalized by characteristic values in the precursor. The characteristic length which represents spontaneous emission is $l_A(=u/A_{21})$, where u is the shock velocity and A_{21} is the spontaneous emission rate.

In order to obtain an analytical solution the exponential integral functions which appear in the radiative flux terms are approximated using the exponential approximation.[12] Thus Eqs. (5) and (6) can be written as

$$\bar{n}_r = \frac{d}{d\eta}[R_g\exp(g\eta)+R_s(\tau_s)] \quad (7)$$

$$\bar{n}_r{}^* = \frac{d}{d\eta}[R_c\exp(c\eta)+R_w\exp(w\eta)-R_s\exp(\tau_s)] - A\beta \quad (8)$$

where R_g, R_s, R_c, R_w and A are defined by comparing Eqs. (7) and (8) to Eqs. (5) and (6).

The optical depths are defined as[1]

$$\tau_g = \frac{l\eta}{l_g\bar{\mu}} = g\eta; \quad \tau_c = \frac{l\eta}{l_c\bar{\gamma}\bar{\mu}} = c\eta; \quad \tau_w = \frac{l\eta}{l_w\bar{\mu}} = w\eta$$

and

$$\tau_s = \frac{l}{l_s\bar{\mu}}\int_0^\eta \beta(y)\,dy = s\int_0^\eta \beta(y)\,dy \quad \text{for} \quad -\infty \leqq \eta \leqq 0$$

For the exponential approximation, $\bar{\mu}$ is taken to be $\frac{2}{3}$. The nondimensional distance in front of the shock is denoted by η. The shock is located at $\eta = 0$ and η goes to $-\infty$ far in front of the shock.

The general equations for the degree of ionization α and the degree of excitation β in the precursor are

$$\frac{d\alpha}{d\eta} = \frac{d}{d\eta}\left\{R_g\exp(g\eta)+R_s\exp\left[s\int_0^\eta \beta(y)\,dy\right]\right\} - C_{ea}\alpha^3 - C_{aa}\alpha^2 \quad (9)$$

and

$$\frac{d\beta}{d\eta} = \frac{d}{d\eta}\left\{R_c\exp(c\eta)+R_w\exp(w\eta)-R_s\exp\left[s\int_0^\eta \beta(y)\,dy\right]\right\} + C_{ea}\alpha^3 + C_{aa}\alpha^2 - C_{ea}{}^*\alpha\beta - C_{aa}{}^*\beta - A\beta \quad (10)$$

subject to the boundary condition that α and β are zero at $\eta = -\infty$. These equations are nonlinear in α and β, consequently, solutions are very difficult to obtain. Each of the radiation terms is attenuated by a different factor which represents the penetration distance of the individual radiative process.

III. Results and Discussion

A. Radiative Structured Precursor Solution

For the radiative structured precursor the collision teams in Eqs. (9) and (10) are omitted. The equations can then be solved analytically to give

$$\alpha(\eta) = R_g\exp(g\eta)+R_{sw}\exp(w\eta)+R_{sc}\exp(c\eta) \quad (11)$$

where

$$R_{sw} = sR_sR_w/(A+w+sR_s) \quad \text{and} \quad R_{sc} = sR_sR_c/(A+c+sR_s)$$

$$\beta(\eta) = \frac{wR_w\exp(w\eta)}{A+w+sR_s} + \frac{cR_c\exp(c\eta)}{A+c+sR_s} \quad (12)$$

$$\tau_s(\eta) = -\frac{sR_w[1-\exp(w\eta)]}{A+w+sR_s} - \frac{sR_c[1-\exp(c\eta)]}{A+c+sR_s} \quad (13)$$

Equation (11) shows that the degree of ionization is increased by the ground state photoionization process and the coupled line excitation followed by the excited state photoionization process. The coupling between the line wings and the excited state is represented by the term R_{sw} and the coupling between the line center and the excited state is represented by the term R_{sc}. The terms R_{sw} and R_{sc} are both dependent on the spontaneous emission; however, both terms are attenuated by the line attenuation factors.

The degree of excitation is increased by the line processes R_w and R_c and reduced by the excited state photionization process, R_s and the spontaneous emission process, A as given by Eq. (12). The degree of excitation attenuates in front of the shock wave due to the line processes. The excited state optical depth is given by Eq. (13). Note that it goes from zero to a maximum of

$$\tau_s(\eta = -\infty) = -s[R_w/(A+w+sR_s)+R_c/(A+c+sR_s)] \quad (14)$$

far in front of the shock. Note that the excited state optical thickness depends upon both the line radiation and the excited state continuum radiation.

B. General Precursor Solution

The general equations as given by Eqs. (9) and (10) are solved to obtain the influence of collisions on α and β in the precursor. The solution is obtained by a successive approximation technique beginning with the radiative solution as the first approximation. The perturbation to the radiation structured precursor is obtained by substituting the radiation solution for α and β into the collisional terms so that they become known functions of η. The equations can then be integrated and the influence of the collisions can be evaluated. The algebra is straightforward and the resulting expressions are very long so they will not be written here. The resulting solution indicates that the collisional terms are negligible in comparison with the radiative terms in the production of α and β.

Table 1 Numerical values for the precursor solutions

M	P(CmHg)	C_{ea}	C_{aa}	C_{ea}*	C_{aa}*	A
13.4	0.157	128.2	0.00573	8.65	3.87×10^{-3}	231.4
18	0.1	38.2	0.00171	4.08	1.83×10^{-4}	172.3
18	1.0	3822	0.171	40.8	1.83×10^{-3}	172.3

M	P(CmHg)	R_g	R_s	R_w	R_c
13.4	0.158	1.1×10^{-11}	9.6×10^{-1}	6.0×10^{-3}	4.1×10^{-9}
18	0.1	9.9×10^{-9}	3.622	3.5×10^{-2}	4.3×10^{-9}
18	1.0	9.2×10^{-9}	3.9×10^{-3}	8.6×10^{-3}	1.8×10^{-9}

M	P(CmHg)	g	s	w	c
13.4	0.158	2.59	1.27×10^{-2}	0.160	2.12×10^{5}
18	0.1	1.64	8.04×10^{-3}	0.101	2.12×10^{5}
18	1.0	16.4	8.04×10^{-2}	1.011	2.12×10^{5}

C. Discussion of Solutions

Table 1 gives typical values of the various parameters for nonequilibrium shock wave structure of an Argon-like gas. Values are shown for three shock waves: 1) Mach 13.4, $P = 0.158$ cm Hg; 2) Mach 18, $P = 0.1$ cm Hg, and 3) Mach 18, $P = 1.0$ cm Hg. For these shock waves most of the radiation is in the excited state region of the spectrum ($\nu_{23} \leqq \nu \leqq \nu_{13}$) so that the free electrons are created by the line wing-excited state continuum process, R_{sw}. From the analytical solution given by Eq. (11) the degree of ionization falls off in front of the shock wave according to the line wing optical depth since the line center radiation is so optically thick that it is completely absorbed at the shock wave and the ground state radiation Rg is small relative to the line wing radiation R_w. The solution of the general equations including the collision term yields the fact that the collisional recombination rate is not effective at reducing the number of free electrons. Thus, the production of free electrons is due entirely to radiative processes.

If one considers the magnitude of the various attenuation factors in Table 1, one sees that ground state photoionization is responsible for the degree of ionization near the shock wave. However, farther away from the shock wave the degree of ionization is determined by the coupled line wing excited state continuum process.

The value of the degree of excitation in the precursor is due entirely to the line wing process and it falls off steadily in front of the shock due to the line wing attenuation. Again, as with α, the numerical solution of the general equation shows that the collisional terms are negligible.

The analytical solution shows the same trends as the numerical solution of Dobbins[7] in that the degree of ionization falls off rapidly just in front of the shock wave and then more slowly farther in front of the shock wave. The degree of excitation falls off rapidly just in front of the shock due to the line center process and then more slowly farther in front of the shock wave due to the line wing process.

The magnitude of the various terms is controlled by the radiative flux which in turn is controlled by the shock wave Mach number and the ambient pressure into which the shock wave is advancing. In general, as the Mach number increases, the ground state photoionization process will increase in importance because of the increase in temperature behind the shock wave. In addition, the radiative lifetime of the excited state and the excited state flux influence the magnitude of both the degree of ionization and the degree of excitation.

IV. Conclusions

This Note presents a very simple analytical solution for shock wave precursors. The analytical solution is used to gain a first-order approximation of influence of collisional processes in the precursor. By examining several shock waves it is shown that 1) the collisional recombination processes are completely negligible for both the free electron and excited state population in the precursor, 2) that the influence of radiative deexcitation through the spontaneous emission process influences both the free electron and excited state population in the precursor, and 3) that far in front of the shock wave one expects the free electron and excited state population to fall off at the same rate; however, their magnitudes will be quite different.

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